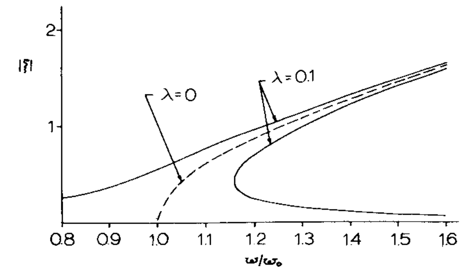

Fig. 1 Nonlinear forced response of a simply supported beam for $k = 1$, the fundamental mode.

is the mass per unit length of the beam. ξ is a dimensionless amplitude parameter defined as

$$\xi = A/(\pi\rho)^{1/2} \tag{25}$$

where A is the maximum amplitude of the lateral deflection and ρ is the radius of gyration of the beam cross section. Since the natural vibration modes are sinusoidal functions of X of the same form as p, it is a straightforward matter to determine the forced response; the result is[7]

$$\xi\left(1 - \frac{\omega^2}{\omega_{ok}^2}\right) + \frac{3\pi}{16}\xi^3 \approx \frac{\lambda}{k^4} \tag{26}$$

λ is a dimensionless loading parameter defined as $Q/(\pi m\rho\omega_{o1}^2)^{1/2}$.

A frequency response curve based upon Eq. (26) for $k = 1$, the fundamental mode, is shown in Fig. 1. The backbone curve for free vibration is also plotted in the figure. The results are typical for a hardening system without damping. Identical results have also been obtained[7] by the method of averaging[9] and by Duffing's method[10]; this strengthens our confidence in the theory as a useful approach for forced vibrations in general.

A related problem is that of the lateral forced vibration of a simply supported rectangular plate with immovable edges. The plate is uniform and Hookean elastic with length a, width b, and thickness h. If the edges are immovable, membrane stresses will be induced in the plate due to transverse flexural vibrations of finite amplitude. For forced vibration in the fundamental mode, we consider a disturbing distributed loading of the form

$$p(X, Y, t) = Q\sin(\pi X/a)\sin(\pi Y/b)\cos\omega t \tag{27}$$

where here p and Q have the units of force per unit area. X, Y are coordinates.

Nonlinear free vibrations are considered in Ref. 2. Forced vibrations in the fundamental mode due to the distributed loading (27) turn out to be governed by[7]

$$\xi\left(1 - \frac{\omega^2}{\omega_o^2}\right) + \xi^3\frac{3}{32(\mu^2+1)^2}\left[\frac{(\mu^4+2\nu\mu^2+1)}{(1-\nu^2)} + \frac{1}{2}(\mu^4+1)\right] \approx \frac{\lambda}{(\mu^2+1)^2} \tag{28}$$

where

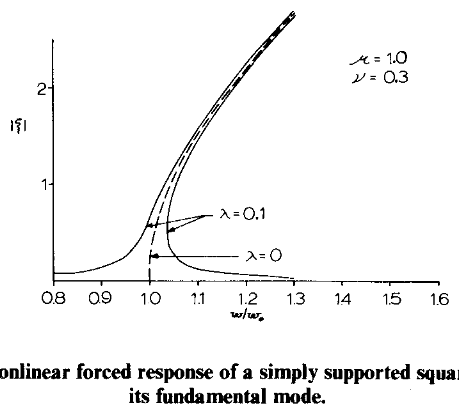

Fig. 2 Nonlinear forced response of a simply supported square plate in its fundamental mode.

$$\xi = [12(1-\nu^2)]^{1/2}\frac{A}{h} \qquad \omega_o^2 = \frac{(\mu^2+1)^2}{12(1-\nu^2)}\frac{Eh^2}{\rho}\left(\frac{\pi}{b}\right)^4$$
$$\lambda = \{[12(1-\nu^2)]^{3/2}/E\}(b/\pi h)^4 Q \tag{29}$$

$\mu = b/a$ is the plate aspect ratio, ν is Poisson's ratio, ρ is the mass density, and A is the maximum amplitude of the lateral deflection.

A frequency response curve based upon Eq. (28) for $\mu = 1$, the square plate, and $\nu = 0.3$ is shown in Fig. 2. The backbone curve for $\lambda = 0$ is also shown in the figure. Again, the results are typical for a hardening system without damping.

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Response of a Three-Layered Ring to an Axisymmetric Impulse

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THE axisymmetric transient response of a three-layered shell having a thick, soft middle layer has received the attention of several investigators. In Ref. 1 solutions were developed for a circular elastic shell supported by an elastic core. This analysis considered propagation and reflection of stress waves in the middle layer or core; and the inner core boundary was specified as a rigid reflector, a free surface, or a support offered by another shell. These solutions were limited to three wave travel times through the elastic core because the algebraic computations became more complicated with each additional wave transit time through the core. This Note presents a modal solution which is not restricted to three wave transit times and which is suitable for studying the stiffening effect of the core and inner shell on the outer shell.

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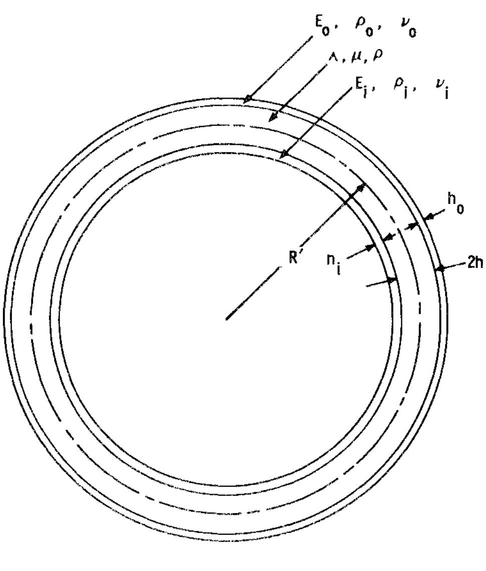

Fig. 1 Geometry of the shell.

Description of the Problem

The three-layered shell shown in Fig. 1 is loaded by an axisymmetric inward impulse of intensity I. Young's modulus, Poisson's ratio, density and thickness of the shell layers are denoted by E, ν, ρ, and h; and the subscripts i and o are used to designate the inner and outer layers, respectively. The density, Lamé's constants, and half-thickness for the central or core layer are denoted by unsubscripted characters ρ, λ, μ, and h. The mean radius of the core layer is R', r' is the radial polar coordinate, t is time and W is the radially outward displacement.

Governing Equations

Elasticity theory is used to analyze the core while shell theory is used to model the inner and outer layers. This implies that the core stiffness is small compared with the other layers. For convenience, the governing equations are written in the dimensionless forms

$$-M_o\frac{\partial^2 u}{\partial \tau^2}(R+1,\tau)-K_o u(R+1,\tau)-(R+1)\frac{\partial u}{\partial r}(R+1,\tau) = U_o\delta(\tau) \tag{1a}$$

$$r\,\partial^2 u/\partial r^2+\partial u/\partial r - r\,\partial^2 u/\partial t^2 = 0, \quad R-1<r<R+1 \tag{1b}$$

$$-M_i\frac{\partial^2 u}{\partial \tau^2}(R-1,\tau)-K_i u(R-1,\tau)+(R-1)\frac{\partial u}{\partial r}(R-1,\tau) = 0 \tag{1c}$$

where

$$u = W/h, \quad R = R'/h, \quad \tau = ct/h, \quad r = r'/h$$

$$U_o = Ic(R+1+h_o/h)/h(\lambda+2\mu), \quad c = [(\lambda+2\mu)/\rho]^{1/2}$$

$$M_i = \frac{\rho_i}{\rho}\frac{h_i}{h}\left(R-1-\frac{h_i}{2h}\right), \quad M_o = \frac{\rho_o}{\rho}\frac{h_o}{h}\left(R+1+\frac{h_o}{2h}\right)$$

$$K_i = \frac{E_i/1-\nu_i^2}{\lambda+2\mu}\,\frac{h_i/h}{R-1-h_i/2h}, \quad K_o = \frac{E_o/1-\nu_o^2}{\lambda+2\mu}\,\frac{h_o/h}{R+1+h_o/2h}$$

Equations (1a) and (1c) govern the inner and outer layers, respectively, and represent boundary conditions on the core displacement u. The K and M coefficients are nondimensional shell stiffness and mass parameters.

Modal Solution

The solution of Eqs. (1) can be written as the sum of modes

$$u(r,\tau) = \sum_n \eta_n(\tau)\phi_n(r) \tag{2}$$

where the η_n are the normal coordinates and the ϕ_n are the mode shapes. However, the standard modal solution technique must be modified in this case because the boundary conditions are time dependent. The method used to solve Eqs. (1) is essentially the same as that used by Ruminer[2] and the details are omitted here. The mode shapes are

$$\phi = Y_o(\omega r)+\alpha J_o(\omega r) \tag{3}$$

where Y_o and J_o are Bessel functions and α is a constant. The admissible ω's (eigenvalues) for this problem are the roots of the characteristic determinant

$$\begin{vmatrix} (M_i\omega^2-K_i)J_o[\omega(R-1)]-(R-1)\omega J_1[\omega(R-1)] & (M_i\omega^2-K_i)Y_o[\omega(R-1)]-(R-1)\omega Y_1[\omega(R-1)] \\ (M_o\omega^2-K_o)J_o[\omega(R+1)]+(R+1)\omega J_1[\omega(R+1)] & (M_o\omega^2-K_o)Y_o[\omega(R+1)]+(R+1)\omega Y_1[\omega(R+1)] \end{vmatrix} = 0 \tag{4}$$

and

$$\alpha = -\frac{(M_i\omega^2-K_i)Y_o[\omega(R-1)]-(R-1)\omega Y_1[\omega(R-1)]}{(M_i\omega^2-K_i)J_o[\omega(R-1)]-(R-1)\omega J_1[\omega(R-1)]} \tag{5}$$

The eigenvalues are ordered such that $\omega_1<\omega_2<\cdots<\omega_n<\cdots$ and the eigenfunction corresponding to an eigenvalue ω_j is denoted by ϕ_j. Following Ref. 2, an orthogonality condition for the mode shapes is derived and used to uncouple the equations for the normal coordinates. The result is

$$d^2\eta_n/d\tau^2+\omega_n^2\eta_n = -(U_o/Q_n)\phi_n(R+1)\,\delta(\tau) \tag{6}$$

where

$$Q_n = M_i\phi_n^2(R-1)+M_o\phi_n^2(R+1)+\int_{R-1}^{R+1} r\phi_n^2\,dr$$

The solution of Eq. (6) satisfying quiescent initial conditions is

$$\eta_n = -[U_o\phi_n(R+1)/Q_n]\sin\omega_n\tau/\omega_n \tag{7}$$

Substituting Eq. (7) into the modal series gives

$$u(r,\tau) = -U_o\sum_{n=1}^{\infty}\frac{\phi_n(R+1)\phi_n(r)}{\omega_n Q_n}\sin\omega_n\tau \tag{8}$$

The strains in the inner and outer shells are

$$\varepsilon_i = u(R-1,\tau)/[R-1-(h_i/2h)] \tag{9a}$$

$$\varepsilon_o = u(R+1,\tau)/[R+1+(h_o/2h)] \tag{9b}$$

and the radial strain in the core is

$$\varepsilon_r = \frac{\partial u}{\partial r} = -U_o\sum_{n=1}^{\infty}\frac{\phi_n(R+1)(d\phi_n/dr)}{\omega_n Q_n}\sin\omega_n\tau \tag{10}$$

Numerical Calculations

Evaluation of the expressions for the strains is straightforward provided the eigenvalues can be found from the transcendental characteristic equation. A simple trial-and-error technique can be used to find the eigenvalues. It was found that the series solutions for the inner and outer shell strains are dominated by their first three or four terms. The solution for the radial strain in the core material is much more slowly convergent, but this problem can be overcome by using the technique of Ref. 3.

For large n, the asymptotic form for a term of the radial strain series is

$$\eta_n\frac{d\phi_n}{dr} \cong \frac{U_o}{M_o}\left(\frac{R+1}{r}\right)^{1/2}\frac{2}{\pi}\frac{\sin(n\pi\tau/2)\cos[n\pi(r-R-1)/2]}{n} \tag{11}$$

This term is identical to the form of the terms in the series representation for the propagating discontinuous step function $f(r)$, where

$$f(r) = \sum_{n=0}^{\infty}\gamma\{H[v\tau+r-R-1-4n]+H[v\tau-r+R+1-4(n+1)]\}$$

$$= \frac{2\gamma}{\pi}\left\{\frac{v\tau}{4}+\sum_{m=1}^{\infty}\sin(m\pi v\tau/2)\cos[m\pi(r-R-1)/2]/m\right\} \tag{12}$$

Matching coefficients of the terms in Eqs. (11) and (12) at the wave front gives the amplitude and nondimensional velocity of the discontinuity

$$v = 1, \quad \gamma = [(R+1)/r_{wf}]^{1/2}U_o/M_o \tag{13}$$

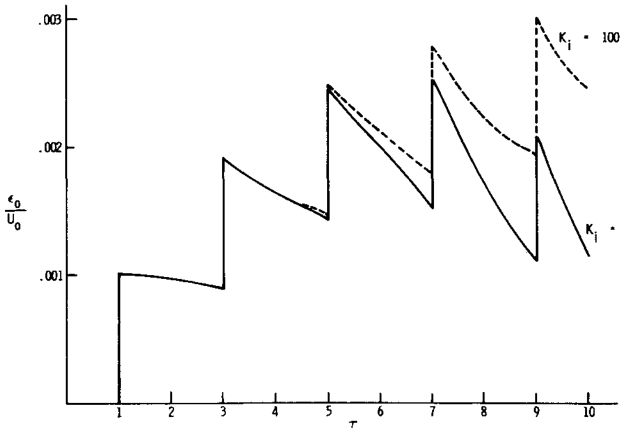

Fig. 2 **Shell strains as a function of K_i for $M_o = 1000$, $M_i = 500$, $K_o = 5$, and $R = 40$.**

where r_{wf} is the location of the wave front at the time for which the series is being evaluated. When Eq. (12) is subtracted from the expression for ε_r, a more rapidly convergent series results

$$\varepsilon_r - f(r) = U_o \left\{ \left[\frac{R+1}{r_{wf}}\right]^{1/2} \frac{\tau}{2\pi M_o} + \sum_{n=1}^{\infty} \left[\frac{\phi_n (R+1)(d\phi_n/dr)}{\omega_n Q_n} \sin \omega_n \tau\right] - \sum_{m=1}^{\infty} \left[\frac{R+1}{r_{wf}}\right]^{1/2} \frac{2}{\pi M_o m} \sin(m\pi\tau/2) \cos\left(m\pi \frac{r-R-1}{2}\right) \right\} \tag{14}$$

Thus ε_r is obtained as the sum of a discontinuous function known in closed form and a rapidly convergent infinite series. Equation (14) is not valid for limiting cases where either M_o or M_i is equal to zero because the asymptotic form in Eq. (11) for $\eta_n (d\phi_n/dr)$ changes. In these cases the procedure used above is still applicable, but a different discontinuous function must be chosen.

Numerical Example

Inspection of Eqs. (1) shows that the governing equations contain five parameters M_o, M_i, K_o, K_i and R. For a numerical example, values of four of the parameters were specified and the fifth, K_i, was allowed to vary. The example was chosen to demonstrate how variations of the inner shell stiffness influence the peak strain in the outer shell. The parameters were chosen as

$$R = 40, \quad M_o = 1000, \quad M_i = 500, \quad K_o = 5, \quad 0.1 < K_i < 1000$$

Figure 2 shows a plot of the peak outer and inner shell strains as a function of K_i. The curves show that for K_i less than 1, the inner shell gives negligible support to the outer shell. For K_i greater than 100, the inner shell is essentially rigid, which means that an increase of the inner shell stiffness has little effect on the amount of support for the outer shell. Figure 3 shows plots of radial strain in the center of the core layer for $K_i = 1$ and $K_i = 100$. These results were obtained by truncating the series in Eq. (14) at $n = 9$ and $m = 7$.

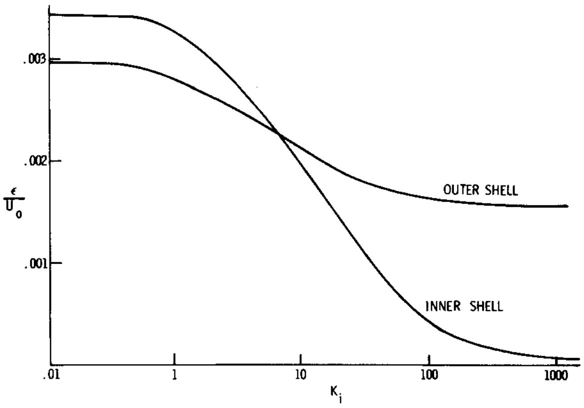

Fig. 3 **Strains at the center of the core for $M_o = 1000$, $M_i = 500$, $K_o = 5$, and $R = 40$.**

Summary and Conclusions

Modal solutions for the strains in a three-layered shell structure have been derived. These solutions are valid for all times, are rapidly convergent, and can easily be programed for evaluation by a computer. By setting $K_i = \infty$ or $K_i = M_i = 0$ the limiting cases of a rigid reflector or no internal shell can be treated. The solution for radial strain in the core contains a propagating discontinuity which, while useful, cannot be expected to give an exact detailed description of the wave front because of the assumption that the shell layers are infinitely rigid in the radial direction.

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A Note on Interacting Boundary Layers

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Nomenclature

$f_1, f_2, \bar{f}_2, f$ = dimensionless stream functions
L = length of moving surface
t = time
T = temperature
u = velocity in x-direction
u_s = shock velocity
U = freestream velocity
x, y = coordinates fixed to the leading edge (A)
$\bar{x}, \bar{y}$ = coordinates fixed to the shock (B) and the trailing edge (B), respectively
α = dimensionless length coordinate
$\eta, \bar{\eta}$ = dimensionless transverse coordinate
μ = dynamic viscosity
ρ = density
ν = kinematic viscosity

Subscript
w = at the wall

Introduction

The problem of two interacting boundary layers, where the equations are singular-parabolic first was considered by Lam and Crocco.[1] They numerically investigated the shock induced boundary layer on a semi-infinite flat plate for infinitely weak shock waves. Further numerical solutions of the boundary-layer equations have been obtained for finite shock strength.[2,3]

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